# **Lesson Objectives**

1. Overview of a Rational Function
2. Describe the Domain of a Rational Function
3. Determine Vertical and/or Horizontal Asymptotes of a Rational Function when given either a formula or a graph

# **Overview** of a Rational Function

**Rational function:** one polynomial divided by another polynomial. Its general format is:

Because of the division, we must remember you **CAN’T** **divide by ZERO!**

If the denominator has a variable, it has the potential to be zero.

# **Domain** of a Rational Function (domain = denominator zero)

|  |  |
| --- | --- |
| * Commandment #1: Thou shalt not **divide by zero**. |  |
| * Rational functions involve dividing with a variable. |
| * We need to find “BAD” values of *x* in denominator that will cause dividing by zero. |
| * **DOMAIN:** set **DENOMINATOR = ZERO**   (numerator typically doesn’t matter for domain\*\*) |

* **EXAMPLE:** Find the domain of the function. Write your answer in set builder notation.

[3.2.75]

DOMAIN = DENOMINATOR ZERO

Solve the equation. (Square root both sides.)

(remember to use after square-rooting)

These are “bad” values for *x*. They cause dividing by zero = “BAD”.   
They must be EXCLUDED!

State the domain. In words: All real numbers EXCEPT and .

In **set-builder notation**:

In interval notation:

**When in doubt – GRAPH IT OUT!!**

Let’s re-visit this function again:

Let’s look at its graph, because it’s very telling.

|  |  |  |
| --- | --- | --- |
|  | Or |  |
| (from [www.desmos.com](http://www.desmos.com)) | (from the graphing calculator) | |

As you move along the function from LEFT to RIGHT, the function BENDS dramatically in two places. It’s like there’s a FORCE FIELD (invisible fence) there. That’s where the function is UNDEFINED, due to dividing by zero.

Remember the domain of this function: In **set-builder notation**:

Those 2 “force-fields” are located exactly in those locations!

The **DOMAIN** restrictions will **ALWAYS** create **UNDEFINED** places in the graph.

There are two ways to be **undefined** in a graph:

* A **vertical asymptote** (vertical line) as is seen above
* A **hole** (open dot) as was seen in a previous section (speed-limit problem)

More on vertical asymptotes and holes a little later…

* **EXAMPLE:** Find the domain of the function. Type your answer in interval notation.

[3.2.83]

DOMAIN = DENOMINATOR ZERO Solve the equation.

Take square root both sides.

**Watch out!**

Can’t square root a negative!

(not possible – not a real number)

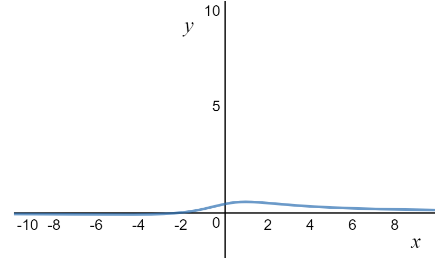
What does this mean?

* There are no “BAD” numbers for the denominator. It will NEVER be zero!
* There is NOTHING to exclude in the domain. All values of *x* will “work.”

(here’s the problem again for reference:)

* **EXAMPLE:** Find the domain of the function. Type your answer in interval notation.

[3.2.83]



State the domain. In words: All real numbers.

In set-builder notation:

In **interval notation**:

**When in doubt – GRAPH IT OUT!!** (graph above from [www.desmos.com](http://www.desmos.com))

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | or |  |  | There are definitely NO breaks in this graph. There are no values to exclude in the domain.  (all real numbers) |
| Hard to see the graph. You can (temporarily) turn the axes off. | Press 2ND, ZOOM (FORMAT) and turn axes off. Then press GRAPH. |

* **EXAMPLE:** Find the domain of the function. Write the answer in set-builder notation.

[3.2.77]

DOMAIN = DENOMINATOR ZERO

Try factoring (it’s easier)

Use Zero Product Property! or

Solve each Equation:

Combine like terms and simplify: or

(here’s the problem again for reference:)

* **EXAMPLE:** Find the domain of the function. Write the answer in set-builder notation.

[3.2.77]

After setting the denominator equal to zero and solving

We have the solutions: or

These are “bad” values for *t*. They cause dividing by zero = “BAD”.

They must be EXCLUDED!

State the domain. In words: All real numbers EXCEPT – 2 and 7.

In **set-builder notation**:

In interval notation:

**When in doubt – GRAPH IT OUT!!**

|  |  |  |  |
| --- | --- | --- | --- |
| (use *x* as your variable on the calculator) | or |  | graph above drawn from [www.desmos.com](http://www.desmos.com) |
| Notice the “force-field” or “invisible fence” at the domain-excluded values  *t* = – 2 and *t* = 6.  These lines are also **VERTICAL ASYMPTOTES**. |

# Determine the **Vertical** and/or **Horizontal Asymptotes**

## **VERTICAL Asymptote (V.A.) - defined**

**Vertical Asymptote (V.A.):** a vertical line that acts as a barrier, or “force-field” for a rational function. The graph of a rational function will NOT cross or pass through a vertical asymptote (V.A.).

On either side near a vertical asymptote (V.A.), the graph of the function will either:

* bend dramatically up
  + (formally , read as “*f*(*x*) approaches positive infinity”)
* bend dramatically down
  + (formally , read as “*f*(*x*) approaches negative infinity”)

A **vertical asymptote** (V.A.) comes from a domain restriction:

* An **undefined** value of the function…
  + Set **denominator equal to ZERO** and solve equation.
* …that doesn’t cancel when factored
  + If numerator has a factor that cancels with denominator, it creates a **“hole,”** not a vertical asymptote.

**Domain**

**Vertical Asymptote (V.A.)**

Doesn’t cancel away

**“Hole”**

It DOES cancel away

A rational function could have exactly one or more than one vertical asymptote (V.A.), or possibly none at all.

## How to find **VERTICAL ASYMPTOTES** (V.A.) of a rational function:

* + - 1. Set denominator equal to zero and solve the equation.
      2. If factored, make sure it doesn’t cancel with factored numerator.
      3. If not, the domain restrictions – each value is a vertical asymptote.

NOTE: There are other ways to have domain restrictions besides setting denominator equal to zero. For example, **square roots** (or other EVEN roots, like 4th root, 6th root, etc.) do not work if the radicand is negative, so you need to account for that by setting **.** Another example is for **logarithms** – they only work if the value (argument of the logarithm) is POSITIVE, so you need to account for that by setting .

(We will not be doing roots or logarithms in this lesson.)

## **HORIZONTAL Asymptote (H.A.) – defined**

Unlike a vertical asymptote that acts like a barrier or “invisible fence”, a horizontal asymptote is different. A rational function *might* cross a horizontal asymptote (H.A.), but it NEVER crosses a vertical asymptote (V.A.).

**Horizontal Asymptote (H.A.):** describes the **end behavior** of *some* rational functions, where BOTH ends “flatten out,” going horizontal.

Extreme LEFT: As (read as: “*x* approaches negative infinity”)

Extreme RIGHT: As (read as: “*x* approaches positive infinity”)

A rational function will have either exactly one horizontal asymptote or none at all.

## How to find **HORIZONTAL ASYMPTOTES** of a rational function:

To find the **horizontal asymptotes** of a basic rational function, you need to **compare** the degree of the numerator to the denominator. One of three things could happen:

|  |  |  |
| --- | --- | --- |
| **Case #1** | **Case #2** | **Case #3** |
|  |  |  |
| **H.A.: NONE – No H.A.** | **H.A.:** | **H.A.:** |
| *Example*: | *Example*: | *Example*: |
|  |  |  |
|  |  |  |
|  |  |  |
| **H.A.: NONE – No H.A.** | **H.A.:** | **H.A.:** or |
| Graph: | Graph: | Graph: |
|  |  |  |
| End behavior: ends NEVER flatten out (no H.A.) | End behavior: ends flatten out along *x*-axis (H.A.: *y* = 0) | End behavior: ends flatten out just below *x*-axis  (H.A.: *y* = -1/3) |

* **EXAMPLE:** Find any horizontal or vertical asymptotes. [4.6.29]

**Vertical Asymptotes (V.A.)**

1. Set denominator equal to zero and solve equation.

Add both sides:

Combine like terms and simplify:

Square root both sides:

Simplify – don’t forget the ±

These are the domain restrictions:

1. Denominator is a difference of squares and factors into .

Here’s the function again:

The denominator doesn’t cancel with numerator.

1. The **vertical asymptotes** (V.A.) are the lines and

**Horizontal Asymptote (H.A.)**

Rewrite with negative in numerator:

Compare the degrees numerator to denominator:

H.A.: The **horizontal asymptote** (H.A.) is the line .

(go on to the next page)

* **EXAMPLE:** Find any horizontal or vertical asymptotes. [4.6.35]

**Horizontal Asymptote** **(H.A.)**

Compare the degrees numerator to denominator:

H.A.: When it’s , there is **NO horizontal asymptote** (H.A.).

**Vertical Asymptotes (V.A.)**

1. Set denominator equal to zero and solve equation.

Factor (it’s the fastest):

Use Zero Product Property: or

Solve each equation:

Combine like terms and simplify: or

These are domain restrictions:

1. Numerator doesn’t factor, so denominator can’t cancel with numerator.
2. The **vertical asymptotes** (V.A.) are the lines and

* **EXAMPLE:** Identify any horizontal or vertical asymptotes in the graph.

State the domain of *f*. [4.6.13]

|  |  |
| --- | --- |
| **Horizontal Asymptote (H.A.):** The end behavior (extreme left and extreme right) of the graph of the function shows that it’s going FLAT along the horizontal line . |  |
| **Vertical Asymptote (V.A.):** The graph of the function bends dramatically along either side of the vertical line . |
| **Domain** of *f*: The vertical asymptote means that it is also an excluded value in the domain – the function is undefined there.  The **domain is** .  (read as: the set of all *x* such that *x* is not equal to 3.) |

* **EXAMPLE:** Identify any horizontal or vertical asymptotes in the graph.

Choose the correct asymptotes below. [4.6.15]

|  |  |
| --- | --- |
| 1. , no horizontal asymptotes |  |
| **B.** |
| **C.** |
| **D.** |
| **SOLUTION:**  **Horizontal Asymptote (H.A.):**  The END BEHAVIOR of the graph of the function shows that it FLATTENS OUT along the horizontal line .  **Vertical Asymptotes (V.A.):** The graph of the function bends dramatically along either side of the vertical lines also written as  The correct answer, therefore, is **D.** | |

Sources Used:

1. MyLab Math for *College Algebra with Modeling and Visualization*, 6th Edition, Rockswold, Pearson Education Inc.
2. Desmos website, <https://www.desmos.com/>, © 2019, Desmos, Inc.
3. Wabbitemu calculator emulator version 1.9.5.21 by Revolution Software, BootFree ©2006-2014 Ben Moody, Rom8x ©2005-2014 Andree Chea. Website <https://archive.codeplex.com/?p=wabbit>